Convergence analysis for Multi-level Spectral Deferred Corrections (MLSDC)

SciCADE 2019

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Motivation

- MLSDC: Multi-level extension of Spectral Deferred Corrections (SDC)
- Several numerical examples indicate its convergence **but** yet no general theoretical proof exists
- Convergence proofs for SDC exist

$\boxed{\Rightarrow \text{Try to use similar ideas to prove MLSDC convergence}}$

Collocation formulation on a single time-step

- Picard form of an initial value problem on $[t_0, t_0 + \Delta t]$

$$u(t) = u_0 + \int_{t_0}^{t} f(u(s))ds$$

- Discretized by spectral quadrature rules with nodes τ_m

$$u_m = u_0 + \Delta t \sum_{j=1}^{M} q_{m,j} f(u_j) \approx u_0 + \int_{t_0}^{\tau_m} f(u(s))ds$$

$$\Longleftrightarrow \underbrace{(I - \Delta t Q F)(U)}_{C(U)} = U_0$$

- Approximation of order $M + 1$

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$$\iff \underbrace{(I - \Delta t Q F)(U)}_{C(U)} = U_0$$

- Approximation of order $M + 1$

$\rightarrow$ How to solve this system efficiently?

Spectral Deferred Corrections (SDC)

A. Dutt, L. Greengard and V. Rokhlin (BIT 2000)

- Standard Richardson iteration ($\hat{=}$ Picard iteration):

$$
\begin{aligned}
U^{(k+1)} &= U^{(k)} + (U_0 - C(U^{(k)})) \\
&= U_0 + (I - \Delta t QF)(U^{(k)})
\end{aligned}
$$

- Preconditioned by use of simpler integration rule Q_Δ:

$$
U^{(k+1)} = U^{(k)} + P^{-1}(U_0 - C(U^{(k)}))
$$

$$
P(U) := (I - \Delta t Q_\Delta F)(U)
$$

$$
\Rightarrow (I - \Delta t Q_\Delta F) U^{(k+1)} = U_0 + \Delta t (Q - Q_\Delta) F(U^{(k)})
$$

- Q_Δ is usually a lower triangular matrix

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Convergence theorem for SDC

T. Tang, H. Xie and X. Yin (J Sci Comput 2012)

Theorem 1 (SDC convergence)

SDC converges linearly with convergence factor $\mathcal{O}(\Delta t)$ to the collocation solution, if Δt is sufficiently small and f is Lipschitz continuous.

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- LTE compared to the solution of the initial value problem is $\mathcal{O}(\Delta t^{\min(k_0+k,M+1)})$
- SDC gains one order per iteration, limited by the number of quadrature nodes
- Order limit $M+1$ stems from collocation problem
  - Higher order for last point in time (e.g. $2M$ for Radau quadrature)

Multi-level SDC (MLSDC)

R. Speck et al. (BIT 2015)

- Multi-level method to solve the collocation problem with SDC iterations on different grids/levels
- Here: Two-grid algorithm (Ω_H: coarse, Ω_h: fine)
- E.g. different resolution in time (number of quadrature nodes M) or space (degrees of freedom N) on the grids
- I_h^H, I_H^h transfer operators (restriction and interpolation)

MLSDC iteration

R. Speck et al. (BIT 2015)

$$\Omega_h : \quad U_h^{(0)} \rightarrow \cdots \rightarrow U_h^{(k)}$$

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MLSDC iteration

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$$\begin{array}{lccc} \Omega_h : & U_h^{(0)} \to \cdots \to U_h^{(k)} \\ & \big\downarrow \text{restriction} \\ \Omega_H : & I_h^H U_h^{(k)} \end{array}$$

MLSDC iteration

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$$
\begin{array}{llcl}
\Omega_h : & U_h^{(0)} \to \cdots \to & U_h^{(k)} & \\
 & & \big\downarrow \text{restriction} & \\
\Omega_H : & & I_h^H U_h^{(k)} & \xrightarrow{\text{SDC with } \tau} U_H^{(k+\frac{1}{2})}
\end{array}
$$

SDC iteration to solve $C_H(U) = U_{0,H} + \tau$ with
$\tau = C_H(I_h^H U_h^{(k)}) - I_h^H C_h(U_h^{(k)})$ and $I_h^H U_h^{(k)}$ as initial guess

MLSDC iteration

R. Speck et al. (BIT 2015)

$$
\begin{tikzcd}
\Omega_h : & U_h^{(0)} \arrow[r] & \cdots \arrow[r] & U_h^{(k)} \arrow[rr] \arrow[d, "\text{restriction}"] & & U_h^{(k+\frac{1}{2})} \\
\Omega_H : & & & I_h^H U_h^{(k)} \arrow[rr, "\text{SDC with } \tau"] & & U_H^{(k+\frac{1}{2})} \arrow[u, "\text{coarse grid correction}"']
\end{tikzcd}
$$

$$U_h^{(k+\frac{1}{2})} = U_h^{(k)} + I_H^h(U_H^{(k+\frac{1}{2})} - I_h^H U_h^{(k)})$$

MLSDC iteration

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SDC iteration to solve $C_h(U) = U_{0,h}$ with $U_h^{(k+\frac{1}{2})}$ as initial guess

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MLSDC iteration

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$$
\begin{array}{lcccccc}
\Omega_h: & U_h^{(0)} \to \cdots \to U_h^{(k)} & \longrightarrow & U_h^{(k+\frac{1}{2})} & \xrightarrow{\text{SDC}} & U_h^{(k+1)} \\
& \Big\downarrow \text{restriction} & & \Big\uparrow \text{coarse grid correction} & & \\
\Omega_H: & I_h^H U_h^{(k)} & \xrightarrow{\text{SDC with } \tau} & U_H^{(k+\frac{1}{2})} & &
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$\boxed{\Rightarrow \text{Does this method converge? How fast?}}$

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Theorem 2 (MLSDC convergence)

MLSDC converges linearly with convergence factor $\mathcal{O}(\Delta t)$ to the collocation solution, if Δt is sufficiently small and f is Lipschitz continuous.

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- MLSDC gains one order per iteration, limited by the number of quadrature nodes
- No improvement compared to SDC $\rightarrow$ Really?

SDC vs. MLSDC convergence

- Upper bound for step size Δt: same
- Comparison of the coefficients: Improvement of MLSDC over SDC seems to depend on $\|(I - I_H^h I_h^H) e_h\|$ with $e_h := U_h - U_h^{(k)}$

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$\rightarrow$ Further analyzed $\|(I - I_H^h I_h^H)e_h\|$

- Assumptions: Coarsening in space with step size Δx, Lagrange interpolation of order p for I_H^h, injection for I_h^H
- If e_h sufficiently smooth:

$$\|(I - I_H^h I_h^H)e_h\| \leq C \Delta x^p \|e_h\|$$

Convergence theorem for MLSDC 2

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Convergence theorem for MLSDC 2

Theorem 3 (improved MLSDC convergence)

MLSDC converges linearly with convergence factor $\mathcal{O}(\Delta t^2)$ to the collocation solution, if Δt is sufficiently small and f is Lipschitz continuous and Δx^p is sufficiently small and $U_h - U_h^{(k)}$ is sufficiently smooth.

Convergence theorem for MLSDC 2

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- LTE compared to the solution of the initial value problem is $\mathcal{O}(\Delta t^{\min(k_0+2k, M+1)})$
- MLSDC can gain two orders per iteration, limited by the number of quadrature nodes
- Order improvement depends on spatial step size Δx, interpolation order p and smoothness of the error

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$\Rightarrow$ Can we see this practically?

Numerical results - Heat equation

- Initial value problem:

$$u_t(x,t) = 0.1u_{xx}(x,t) \quad \forall t \in [0, \Delta t],\ x \in [0,1],$$
$$u(0,t) = 0, \quad u(1,t) = 0,$$
$$u(x,0) = sin(4\pi x)$$

- Analytical solution known
- Method parameters:
  - Transformed to ODE by finite-difference method
  - M = 5 quadrature nodes
  - Q_Δ corresponds to right-hand rectangle rule (implicit Euler)

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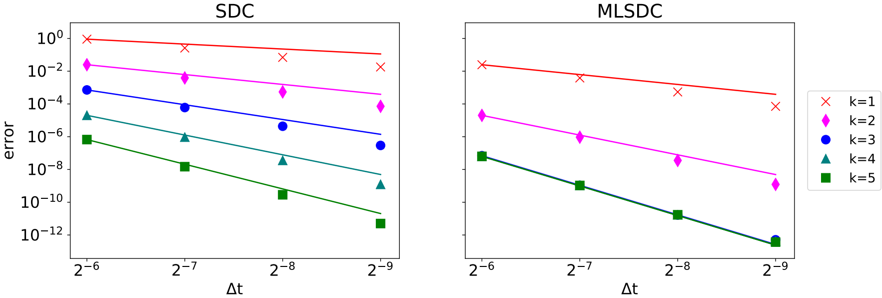
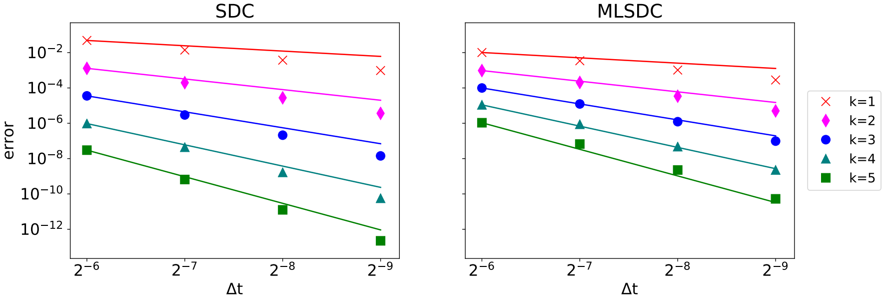


SDC (Theorem 1): LTE $= \mathcal{O}(\Delta t^{\min(k_0+k, M+1)})$

MLSDC (Theorem 3): LTE $= \mathcal{O}(\Delta t^{\min(k_0+2k, M+1)})$

Numerical results - Heat equation

- $\Delta x = 2^{-4}$, $p = 8$, spread initial value as initial guess (smooth)

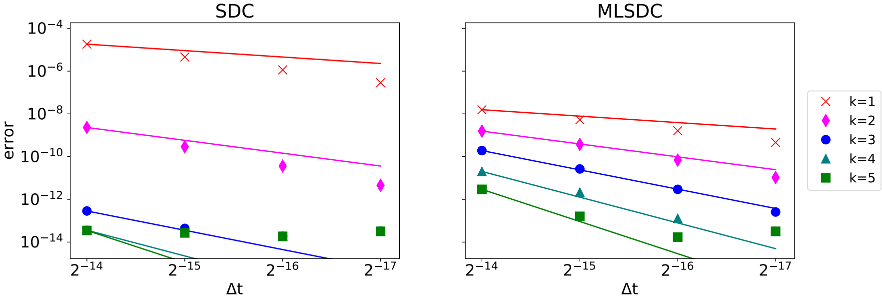


$$\text{SDC (Theorem 1): LTE} = \mathcal{O}(\Delta t^{\min(k_0+k, M+1)})$$

$$\text{MLSDC (Theorem 2): LTE} = \mathcal{O}(\Delta t^{\min(k_0+k, M+1)})$$

Numerical results - Heat equation

- $\Delta x = 2^{-8}$, $p = 4$, spread initial value as initial guess (smooth)



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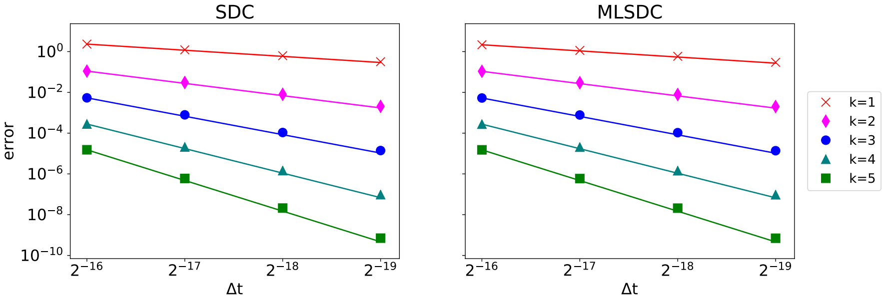
Numerical results - Heat equation

- $\Delta x = 2^{-8}$, $p = 8$, random initial guess (not smooth)



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Conclusion and outlook

Summary

- Theoretical proof for MLSDC convergence
- MLSDC gains one or two orders per iteration, limited by the number of quadrature nodes (if Δt small and f Lipschitz continuous)
  $\rightarrow$ Conditions for higher order: Δx small, p high, error smooth

Conclusion and outlook

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What's next?

- Further analysis of conditions for the smoothness of the error
- Use convergence results to construct a time-adaptive method
- Convergence analysis for other extensions of SDC (e.g. SISDC)